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On space–times admitting a three-parameter isometry group with two-dimensional null orbits

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Abstract. Space–times admitting a three-parameter group of motions acting on two-dimensional null orbits are considered. A coordinate transformation relating the canonical form of the metrics as given by Petrov and the apparently more general form given by Defrise is exhibited. For pure radiation fields the space–time is a PP wave. The general vacuum solution with a cosmological constant is found in closed form.

1. The metric

In his book, Petrov (1969) listed the canonical forms of the metrics of space–times admitting isometry groups. Amongst these he gave the metric admitting a three-parameter group of motions of Bianchi type 2 acting on two-dimensional null orbits. More recently, Defrise (1969) and Goenner and Stachel (1970) have suggested that Petrov's metric is not sufficiently general and proposed a canonical form involving an additional arbitrary function of one variable.

The two forms of the metric are, however, equivalent and the additional function may be eliminated by means of a coordinate transformation. To demonstrate this I take as a starting point the Killing vectors given by Defrise (1969):

$$X_0 = \partial/\partial x^0 \qquad X_\alpha = B_\alpha(x^3)x^1(\partial/\partial x^0) + A_\alpha(x^3)(\partial/\partial x^1) \tag{1}$$

where $\alpha = 1, 2$ and $\boldsymbol{A} \times \boldsymbol{B} = 1$. Here and below it is convenient to use two-dimensional vector notation, i.e. $\boldsymbol{A} = (A_1, A_2)$, etc.

From Killing's equations one can deduce that the most general metric admitting the three Killing vectors of equation (1) is

$$\begin{aligned} ds^2 = \alpha^2(x^2, x^3)\{&-2\, dx^0\, dx^3 + \lambda(x^3)(dx^1)^2 + 2\rho(x^3)x^1\, dx^1\, dx^3 \\ &+ [\sigma(x^3)(x^1)^2 + 2\epsilon(x^2, x^3)](dx^3)^2\} \\ &+ \beta^2(x^2, x^3)(dx^2)^2 + 2\gamma(x^2, x^3)\, dx^2\, dx^3 \end{aligned} \tag{2}$$

where

$$\lambda(x^3) = (\boldsymbol{A} \times \dot{\boldsymbol{A}})^{-1} \qquad \rho(x^3) = \lambda(\boldsymbol{B} \times \dot{\boldsymbol{A}}) \qquad \sigma(x^3) = (\dot{\boldsymbol{B}} - \rho\dot{\boldsymbol{A}}) \times \boldsymbol{B} \tag{3}$$

and α, β, γ and ϵ are arbitrary functions of x^2 and x^3. A prime and a dot denote differentiation with respect to x^2 and x^3 respectively.

Not all the functions appearing in the metric are essential and may be eliminated by means of coordinate transformations preserving the form of the Killing vectors in (1). The metric given by Petrov is obtained by setting $\sigma = \rho = \lambda - 1 = \epsilon = \gamma = 0$ in (2). From (3) it follows that $\dot{\boldsymbol{A}} = \boldsymbol{B}$ and $\dot{\boldsymbol{B}} = 0$ and the basis of the Lie algebra can be chosen so that the Killing vectors take the form given by Petrov:

$$X_0 = \partial/\partial x^0 \qquad X_1 = \partial/\partial x^1 \qquad X_2 = x^1(\partial/\partial x^0) + x^3(\partial/\partial x^1). \tag{4}$$

Defrise's form of the metric is obtained by setting $\rho = \lambda - 1 = \gamma = 0$ and the components of the Killing vectors satisfy the equations $\dot{\boldsymbol{A}} = \boldsymbol{B}$ and $\dot{\boldsymbol{B}} = \sigma\boldsymbol{A}$. Goenner and Stachel give the same Killing vectors as Defrise but the metric function ϵ is set equal to zero in addition to ρ, $\lambda - 1$ and γ.

A somewhat tedious calculation reveals that the most general coordinate transformation preserving the form of (1) is

$$\tilde{x}^0 = x^0 + f(x^3)(x^1)^2 + g(x^2, x^3) \tag{5a}$$

$$\tilde{x}^1 = h(x^3)x^1 \tag{5b}$$

$$\tilde{x}^2 = k(x^2, x^3) \tag{5c}$$

$$\tilde{x}^3 = l(x^3) \tag{5d}$$

where f, g, h, k and l are arbitrary functions of their arguments subject only to the restrictions that $\dot{l}$, k' and h are all non-zero. Under these transformations the Killing vectors and metric change as follows:

$$\tilde{\boldsymbol{A}} = h\boldsymbol{A} \tag{6a}$$

$$\tilde{\boldsymbol{B}} = (\boldsymbol{B} + 2f\boldsymbol{A})/h \tag{6b}$$

$$\tilde{\alpha}^2 = \alpha^2/\dot{l} \tag{6c}$$

$$\tilde{\lambda} = \lambda\dot{l}/h^2 \tag{6d}$$

$$\tilde{\rho} = h^{-2}(\rho - \lambda h^{-1}\dot{h} + 2f) \tag{6e}$$

$$\tilde{\sigma} = (h^2\dot{l})^{-1}(\sigma - 2\tilde{\rho}h\dot{h} + \tilde{\lambda}\dot{h}^2/\dot{l} + 2\dot{f}) \tag{6f}$$

$$\tilde{\beta} = \beta/k' \tag{6g}$$

$$2\tilde{\gamma} = (k'\dot{l})^{-1}(2\gamma - \beta^2\dot{k}/k' - \alpha^2 g') \tag{6h}$$

$$\tilde{\epsilon} = (\epsilon - \alpha^{-2}\tilde{\gamma}\dot{k}\dot{l} - \tfrac{1}{2}\alpha^{-2}\tilde{\beta}^2\dot{k}^2 - \dot{g})/\dot{l}. \tag{6i}$$

If l, h and f are chosen to satisfy the ordinary differential equations

$$\dot{l} = h^2/\lambda \qquad \dot{h} = h(\rho + 2f)/\lambda \qquad 2\dot{f} = (\rho + 2f)^2/\lambda - \sigma \tag{7}$$

then $\tilde{\lambda} = 1$ and $\tilde{\rho} = \tilde{\sigma} = 0$.

The additional function $\sigma(x^3)$ in Defrise's metric has been eliminated. By a suitable choice of the functions g and k the metric components $\tilde{\epsilon}$ and $\tilde{\gamma}$ may be set to zero. To see this, following Petrov (1969) we put $\tilde{\epsilon} = \tilde{\gamma} = 0$ in ($6h$, i) and obtain partial differential equations of Cauchy–Kowalewski type for $\dot{k}$ and $\dot{g}$. Thus the equivalence of the two canonical forms has been demonstrated.

The functions l, h and f are not completely determined by (7). However, the remaining freedom cannot be used to simplify the metric but merely reflects the freedom of choice in the basis of a Bianchi type-2 algebra. Given any constants a, b and c such that $bc \neq 0$ the coordinate x^1 may be chosen so that $aX_0 + bX_1 + cX_2 = \partial/\partial x^1$.

As indicated above, g and k may be chosen so that $\tilde{\epsilon} = \tilde{\gamma} = 0$. However, it will be more convenient below to choose these two functions so that $\tilde{\beta} = 1$ and $\tilde{\gamma} = 0$. This is achieved if k and g satisfy the equations

$$k' = \beta \qquad \alpha^2 g' = 2\gamma - \beta\dot{k}. \tag{8}$$

2. Pure radiation fields

I will henceforth use the following metric form which was derived in the previous section:

$$\mathrm{d}s^2 = \alpha^2(x^2, x^3)\{-2\,\mathrm{d}x^0\,\mathrm{d}x^3 + (\mathrm{d}x^1)^2 + 2\epsilon(x^2, x^3)(\mathrm{d}x^3)^2\} + (\mathrm{d}x^2)^2. \tag{9}$$

It will be convenient at this point to introduce a pseudo-orthonormal tetrad of one-forms and a two-form whose components are given by

$$\begin{aligned} X_{0i} = l_i = \alpha^2\delta_i^3 \qquad & \underset{1}{e}_i = \alpha\delta_i^1 \qquad \underset{2}{e}_i = \delta_i^2 \\ n_i = -\delta_i^0 + \epsilon\delta_i^3 \qquad & w_{ij} = \sqrt{2}\,l_{[i}\{\underset{1}{e}_{j]} + \mathrm{i}\underset{2}{e}_{j]}\}. \end{aligned} \tag{10}$$

The Ricci tensor and complex self-dual Weyl tensor of the metric (9) can then be conveniently written as

$$R_{ij} = \alpha^{-2}(\alpha\alpha'' + 2\alpha'^2)\{2l_{(i}n_{j)} + \underset{1}{e}_i\underset{2}{e}_j\} + 3\alpha^{-1}\alpha''\underset{2}{e}_i\underset{2}{e}_j + 4\alpha^{-4}(\alpha\dot{\alpha}' - \dot{\alpha}\alpha')l_{(i}\underset{2}{e}_{j)} + Cl_il_j \tag{11a}$$

$$\overset{+}{C}_{ijkl} = Dw_{ij}w_{kl} \tag{11b}$$

where

$$C = \alpha^{-6}(\alpha\ddot{\alpha} - 2\dot{\alpha}^2) + \alpha^{-3}(\alpha\epsilon'' + 3\alpha'\epsilon') \tag{11c}$$

$$D = \alpha^{-6}(\alpha\ddot{\alpha} - 2\dot{\alpha}^2) - \alpha^{-3}(\alpha\epsilon'' + \alpha'\epsilon'). \tag{11d}$$

Clearly l^i is both a Ricci eigenvector and the principal null vector of the Weyl tensor. The Weyl tensor is type N or zero: a fact which could have been foreseen as the isotropy group of the space–time contains the null rotations about l^i (Ehlers and Kundt 1962).

The following theorem which generalises that of the author (Barnes 1973) will now be proved.

Theorem. A space–time which admits a three-parameter group of motions acting multiply transitively on two-dimensional null orbits and which is a pure radiation field is a plane-fronted wave with parallel rays (PP wave). Further, if the space–time is a vacuum or an invariant Einstein–Maxwell field, it is a plane-wave space–time.

Proof. For pure radiation, $R_{ab} = Ek_ak_b$ for some scalar E and some null vector k_a. This is compatible with (11a) if and only if k_a is proportional to l_a and $\alpha' = 0$. It may readily be verified that the latter condition implies that both l_i and w_{ij} are covariantly constant. It follows immediately from a result of Ehlers and Kundt (1962) that the field is a PP wave. As D is real, it also follows that the polarisation of the gravitational wave is a constant (equal to 0 or π).

By definition the wave is plane if and only if its amplitude D is constant in the wavefronts (i.e. $D = D(x^3)$). This is equivalent to the condition $C = C(x^3)$ as, if $\alpha' = 0$, C and $-D$ differ only by a function of x^3. For a vacuum field $C = 0$ and so the second part of the theorem follows trivially.

A source-free Einstein–Maxwell field is necessarily null with $\mathbf{l}'$ as its principal null vector since, otherwise, the Ricci tensor would be incompatible with $(11a)$. Hence the Maxwell bivector can be written in the form

$$F_{ij} = 2Fl_{[i}\underset{1}{e}_{j]} + 2Gl_{[i}\underset{2}{e}_{j]} \tag{12}$$

where $C = F^2 + G^2$. From Maxwell's equations it follows that

$$F + \mathrm{i}G = H[x^3]\exp[J(x^3)(x^2 + \mathrm{i}\alpha(x^3)x^1) + \mathrm{i}\phi(x^3)] \tag{13}$$

where H, J and ϕ are arbitrary functions of integration. If the Maxwell bivector is invariant under the action of the isometry group then $L_{X_\alpha}F_{ij} = 0$, where the X_α's are given by (4). It follows that $F_{ij,1} = 0$ and consequently $J(x^3) = 0$. Thus $C = H^2(x^3)$ and D are functions of x^3 alone and both the electromagnetic and gravitational fields are plane waves.

At this stage it is worth completing the solution for a general Einstein–Maxwell field as it has the somewhat unusual feature that the gravitational field is invariant under the action of the isometry group whereas its source F_{ij} given by (12) and (13) is not invariant unless $J = 0$. The field equations are

$$R_{ij} = H^2(x^3)\exp(2J(x^3)x^2)l_il_j \tag{14}$$

where R_{ij} is given by $(11a, c)$. If $J \neq 0$, the general solution is

$$2\epsilon = \frac{\alpha^2(x^3)H^2(x^3)}{2J^2(x^3)}\exp(2J(x^3)x^2) - \frac{1}{\alpha^3}\left(\ddot{\alpha} - \frac{2\dot{\alpha}^2}{\alpha}\right)(x^2)^2 \tag{15}$$

where α, H and J are arbitrary functions of x^3. The two functions of integration appearing in ϵ have been removed by means of the coordinate transformation

$$\tilde{x}^0 = x^0 + \alpha^{-2}\dot{K}(x^3)x^2 + L(x^3) \qquad \tilde{x}^1 = x^1 \qquad \tilde{x}^2 = x^2 + K(x^3) \qquad \tilde{x}^3 = x^3 \tag{16}$$

where K and L are suitably chosen functions of x^3.

3. Vacuum fields with a cosmological constant

In this section the general solution is found for which the Ricci tensor is given by $R_{ij} = Kg_{ij}$ where K is the cosmological constant. From $(11a, c)$ it follows that

$$\alpha = \alpha_0(x^3)\exp(cx^2) \qquad \epsilon'' + 3c\epsilon' = \alpha_0^{-4}(2\dot{\alpha}_0^2 - \alpha_0\ddot{\alpha}_0)\exp(-2cx^2) \tag{17}$$

where α_0 is an arbitrary function of x^3 and $3c^2 = K$. Integrating the latter equation one obtains

$$\epsilon = \gamma_0(x^3) + \epsilon_0(x^3)\exp(-3cx^2) + (2c^2\alpha_0^4)^{-1}(\alpha_0\ddot{\alpha}_0 - 2\dot{\alpha}_0^2)\exp(-2cx^2) \tag{18}$$

where ϵ_0 and γ_0 are arbitrary functions of integration.

The coordinate freedom preserving the form of the metric (9) includes the transformation

$$\tilde{x}^0 = x^0 - (2c\alpha_0^2)^{-1}\dot{K}\exp(-2cx^2) + L(x^3)$$
$$\tilde{x}^1 = hx^1 \qquad \tilde{x}^2 = x^2 + K(x^3) \qquad \tilde{x}^3 = hx^3 + k \tag{19}$$

where h and k are constants. Under this transformation

$$\tilde{\epsilon} = \epsilon + [(2c\alpha_0^2)^{-1}\ddot{K} - (2\alpha_0^2)^{-1}\dot{K}^2 - \alpha_0^{-3}\dot{K}\dot{\alpha}_0]\exp[2c(K - \tilde{x}^2)] - L.$$

By a suitable choice of K and L one may put (omitting tildes)

$$\epsilon = \epsilon_0(x^3)\exp(-3cx^2) \qquad \alpha\ddot{\alpha}_0 - 2\dot{\alpha}_0^2 = 0. \tag{20}$$

The constants h and k in (19) may be chosen so that, without loss of generality, $\alpha_0 = (x^3)^{-1}$.

Thus the general vacuum solution with cosmological constant is given by the metric (9) with

$$\alpha = \exp(cx^2)/x^3 \qquad \epsilon = \epsilon_0(x^3)\exp(-3cx^2). \tag{21}$$

This field is not a PP wave as $\alpha' \neq 0$. However, the null vector l^i is hypersurface orthogonal, shear- and divergence-free and geodesic and hence the field is a plane-fronted wave (Ehlers and Kundt 1962).

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